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Enumeration of almost polynomial rational functions with given critical values

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Abstract

Enumerating ramified coverings of the sphere with fixed ramification types is a well-known problem first considered by Hurwitz [A. Hurwitz, Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten, Mathematische Annalen 39 (1891) 1–61. [7]]. Up to now, explicit solutions have been obtained only for some families of ramified coverings, for instance, those realized by polynomials in one complex variable. In this paper we obtain an explicit answer for a large new family of coverings, namely, the coverings realized by simple almost polynomials, defined below. Unlike most other results in the field, our formula is obtained by elementary methods.

1. Rational functions and minimal factorizations of permutations

Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function of degree n in one complex variable. A *critical point* of f is a point $z \in \mathbb{C}$ such that $f'(z) = 0$. Its *degree* is the number $a \geq 2$ such that f looks like $f(z) = z^a$ in the neighborhood of the critical point. A *critical value* of f is its value at a critical point. (Note that we do not count poles as critical points.)

Definition 1.1. A rational function f is called *simple* if every critical value of f has exactly one critical preimage. It is called *almost polynomial* if the sum of orders of its poles is smaller than the degree of each critical point.

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In particular, an almost polynomial rational function has a numerator of a much bigger degree than the denominator.

Our goal is to find the number of simple almost polynomial rational functions with fixed orders of poles and fixed critical values of fixed degrees, a problem first considered by Arnold [1].

The general problem of enumerating ramified coverings of the sphere with fixed ramification types can, in some sense, be solved using the representation theory of the symmetric group; however the answer one obtains is a rather complicated sum over the irreducible representations. In particular, there is still no simple criterion for determining whether the number of coverings is equal to 0 or not.

For (not necessarily simple) polynomials the problem can be reduced to a combinatorial problem solved by Goulden and Jackson in [3]. Later, when the relation to polynomials was discovered, their formula was reproved in [9,10,8] (appendix by D. Zagier) by three different methods. In our paper [10] we also treated the case of almost polynomial functions with just one simple pole. In all these cases, as well as in the case considered in the present note, one obtains simple closed answers.

Goupil and Schaeffer [6] generalized Goulden and Jackson's result on polynomials to meromorphic functions with a unique pole on Riemann surfaces of any genus, but their answer is not as explicit. For other results on the enumeration of ramified coverings and their relation to the intersection theory on moduli spaces see [2,4,5,11] and the references therein.

All rational functions we consider in the following are simple and almost polynomial.

A rational function f can be viewed as a ramified covering of the Riemann sphere by the Riemann sphere. Going around a critical value in the image we obtain a permutation of the sheets in the preimage (the monodromy of the covering). It follows from Riemann's existence theorem that the problem of counting rational functions can be reformulated in terms of permutations (for details see [8], Chapter 2).

Let $\sigma \in S_n$ be a permutation of $n = \deg f$ elements.

Definition 1.2. A product $\sigma_k \dots \sigma_1 = \sigma$ is called a *minimal factorization* of σ if (i) the group generated by $\sigma_1, \dots, \sigma_k$ acts transitively on the set $\{1, \dots, n\}$ and (ii) the total number of cycles in the permutations $\sigma_1, \dots, \sigma_k, \sigma$ equals $kn - n + 2$.

Here σ corresponds to the monodromy of f at ∞, while the σ_j's are the monodromies at the critical values. The conditions guarantee that the ramified covering determined by the permutations $\sigma_1, \dots, \sigma_k, \sigma$ is (i) connected and (ii) of genus 0 (by the Riemann–Hurwitz formula). Rather than counting the cycles of the permutations, it is more natural to consider their *defects*: a defect being n—(the number of cycles). If the Euler characteristic of the covering surface equals χ, the sum of defects of the corresponding monodromies equals $2n-\chi$. Hence the total defect $2n-2$ of a minimal factorization is the smallest possible for a transitive factorization, which explains the word "minimal".

For brevity, we will say that *a rational function f is of type* $(a_1, \dots, a_k)$, $(p_1, \dots, p_c)$ if it has c poles of orders $p_1, \dots, p_c$ and k critical points of degrees $a_1, \dots, a_k$. Since f is almost polynomial, the sum $p = p_1 + \dots + p_c$ satisfies $p < a_j$ for all j. Similarly, we say that *a minimal factorization is of type* $(a_1, \dots, a_k)$, $(p_1, \dots, p_c)$ if σ has cycles of lengths $p_1, \dots, p_c, n-p$, and each σ_j is an a_j-cycle, *i.e.*, has exactly one cycle of length $a_j \geq 2$, the other points being fixed.

To a minimal factorization of type $(a_1, \dots, a_k)$, $(p_1, \dots, p_c)$ we assign a colored graph with oriented edges called a *constellation*. It is obtained in the following way. Take n numbered vertices. For each j, $1 \leq j \leq k$ form an oriented polygon using the vertices from the cycle of σ_j. The edges of the polygon are colored with "color" j. Now forget the numbers of the vertices.

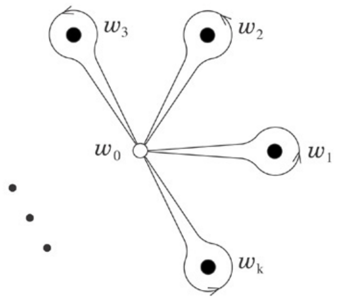


Fig. 1. A star of loops.

It is clear that the constellation allows us to reconstitute the permutations $\sigma_1, \dots, \sigma_k, \sigma$ up to a common conjugation.

Definition 1.3. A *constellation* is a connected graph, whose edges are oriented and colored with colors from 1 to k, obtained by gluing together k oriented polygons with colors $1, \dots, k$ at some of their vertices. A vertex of a polygon cannot be glued to another vertex of the same polygon.

Summarizing, we now have three equivalent problems: given $(a_1, \dots, a_k)$, $(p_1, \dots, p_c)$, count the number of rational functions, minimal factorizations, and constellations of this type. The three numbers are related by simple combinatorial multiplicative factors.

Before proceeding, let us recall some basic facts about constellations applied to our case (for more details see [8]).

Denote by $w_1, \dots, w_k$ the critical values of f. Choose any complex number w_0 different from $w_1, \dots, w_k$ and draw on the plane a family of k loops isotopic to the star of loops in Fig. 1. To f we assign a constellation in the following way.

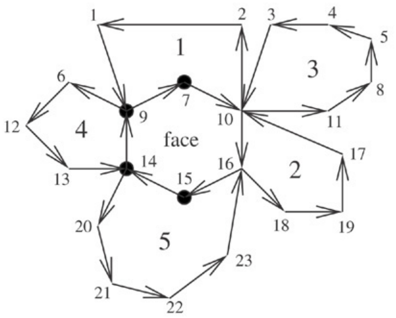
The point w_0 has n simple preimages under f; they will be the vertices of the constellation. The preimage of the loop l_j, surrounding w_j, is a union of n arcs going from one preimage of w_0 to another. Among these arcs, a_j form an oriented cycle, while $n - a_j$ others are closed loops. Erase the $n - a_j$ closed loops, but keep the a_j arcs of the cycle and color them with color j. We have obtained the constellation assigned to f.

Once the critical values $w_1, \dots, w_k$ are fixed, the constellations are in one-to-one correspondence with the ramified coverings f. However, this one-to-one correspondence is not canonical, but depends on the choice of the loops up to isotopy.

Note that we have constructed the constellation together with an embedding into a sphere. However, the embedding carries no new information, because it can be reconstructed uniquely up to isotopy from the combinatorial structure of the constellation. Namely, the embedding is determined by the following conditions: (i) The orientations of the edges determine the counterclockwise orientation on each polygon; (ii) There are no vertices inside the region surrounded by a polygon; (iii) If we choose any vertex and enumerate the colors of the polygons surrounding it in the counterclockwise order starting from the smallest color we obtain an increasing sequence of colors.

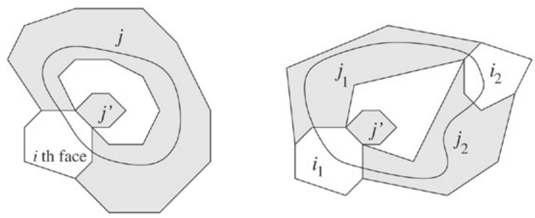
From now on we will assume that the constellations are embedded into a sphere.

Cutting the sphere along the edges of the embedded constellation we obtain $k + c + 1$ pieces homeomorphic to open discs. Among them, k correspond to the polygons and the $c + 1$ others to the cycles of σ. The piece corresponding to the long cycle (of length $n - p$) will be called the *exterior face*; the other c pieces will be called *interior faces* or just *faces*. Note that not all



$$\sigma_1 = (1, 9, 7, 10, 2)\,,$$
$$\sigma_2 = (10, 16, 18, 19, 17)\,,$$
$$\sigma_3 = (3, 10, 11, 8, 5, 4)\,,$$
$$\sigma_4 = (6, 12, 13, 14, 9)\,,$$
$$\sigma_5 = (14, 20, 21, 22, 23, 16, 15)\,.$$

Fig. 2. A constellation with numbered vertices and the corresponding permutations. The color of each polygon is marked inside it. The essential vertices of the unique face are shown in black; these vertices form a cycle of $\sigma_5 \ldots \sigma_1$.



(A) (B)

Fig. 3. The polygon j' should have an edge on the exterior face.

vertices of a face are contained in the corresponding cycle of σ. Those which are will be called *essential vertices*, see Fig. 2.

A vertex of a face is essential if and only if the color of the edge that follows it in the clockwise direction is smaller than or equal to the color of the edge that precedes it. Every vertex is essential for exactly one (interior or exterior) face, namely, the face lying between the polygons with biggest and smallest colors among those meeting at the vertex.

Finally, let us explain the meaning of the condition $p < a_j$, specific to the problem we consider.

Lemma 1.4. *The condition $p < a_j$ implies that the jth polygon has an edge on the exterior face.*

Proof. In every face, there is at least one essential vertex between two edges of the same color. Thus, if the ith face contains m_{ij} edges of color j, we have $p_i \geq m_{ij}$. Assuming that the jth polygon does not have an edge on the exterior face, we obtain $p = \sum p_i \geq \sum_i m_{ij} = a_j$, which contradicts the condition $p < a_j$. $\diamond$

Now we assume that $p < a_j$ for all j.

Lemma 1.5. *If a polygon has common edges with a face then these edges are consecutive both on the face and on the polygon.*

Proof. If the boundary of the ith face contains two intervals of color j, then some another polygon j' can be surrounded by a closed line entirely contained in the ith face and the jth polygon (Fig. 3(A)). But this is impossible, because the polygon of color j' has an edge on the exterior face.

Once we know that the edges are consecutive on the face, they are automatically consecutive on the polygon, because two vertices of the same polygon cannot be glued together in a constellation. $\diamond$

Thus it makes sense to talk about the cyclic order of polygons forming a face, about two consecutive polygons at a face, and so on.

Lemma 1.6. *If polygons j_1 and j_2 have common edges with faces i_1 and i_2, then the polygons j_1 and j_2 are consecutive at both faces.*

Proof. The proof is similar to the previous one: if the polygons j_1 and j_2 are not consecutive on one of the faces, then some other polygon j' is surrounded by a closed line lying entirely in the polygons j_1, j_2 and in the faces i_1, i_2 (Fig. 3(B)). $\diamond$

2. The main theorem

Let $p_1, \ldots, p_c$ be $c \geq 0$ positive integers (orders of poles), $\sum p_i = p$. Fix an integer $n > p$ (degree of the almost polynomial). Let $a_1, \ldots a_k$ be $k \geq 2$ more positive integers (multiplicities of the critical points) satisfying $a_j > p$ for all j and $\sum a_j = n + k + c - 1$ (the Riemann–Hurwitz formula). Denote by $|\mathrm{Aut}\{p_1, \ldots, p_c\}|$ the number of permutations s of c elements such that $p_i = p_{s(i)}$ for all i. For instance, $|\mathrm{Aut}\{4, 4, 3, 2, 2, 2, 2, 2, 1, 1\}| = 2! \cdot 1! \cdot 5! \cdot 2!$.

Consider a permutation $\sigma \in S_n$ with cycle type $(p_1, \ldots, p_c, n - p)$.

Theorem 1. *The number of minimal factorizations of σ into a_j-cycles equals*

$$\frac{(k+c-2)!}{(k-2)!}\, p_1^2 \ldots p_c^2\, (n-p)^{k-1}.$$

The number of constellations as well as the number of rational functions with fixed critical values of type $(a_1, \ldots, a_k)$, $(p_1, \ldots, p_c)$ equals

$$\frac{1}{|\mathrm{Aut}\{p_1, \ldots, p_c\}|} \frac{(k+c-2)!}{(k-2)!}\, p_1 \ldots p_c\, (n-p)^{k-2}.$$

The three assertions of the theorem are equivalent. Indeed, the number of constellations and the number of rational functions of a given type coincide, which follows from Riemann's existence theorem (see [8] for more details). On the other hand, to obtain a minimal factorization of σ from a given constellation, we must number the vertices of the constellation in such a way that the product $\sigma_k \ldots \sigma_1$ equals σ. Once we have found one such numbering, the others are obtained by the action of the stabilizer of σ in S_n. Thus there are

$$p_1 \ldots p_c(n-p) \cdot |\mathrm{Aut}\{p_1, \ldots p_c\}|$$

ways of numbering the vertices: this is the number of permutations that commute with σ.

In the next two sections we prove the theorem for constellations.

3. Assembling a constellation

We are going to prove the assertion of the theorem on the number of constellations. We start by labeling the faces of the constellations so as to make them distinguishable, which kills the

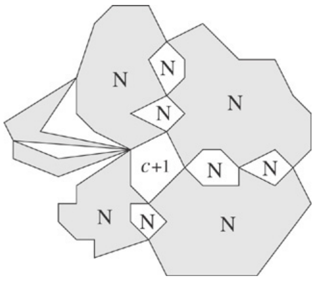


Fig. 4. The $(c+1)$st face in this figure has 4 neighboring polygons and 5 neighboring faces, marked with "N".

|Aut| factor. We must show that the number of such constellations with labeled faces equals

$$\frac{(k+c-2)!}{(k-2)!}\, p_1 \ldots p_c\, (n-p)^{k-2}.$$

We proceed by induction on the number c of faces in the constellations.

For $c = 0$, the constellations have no (interior) faces, *i.e.*, they are "trees" glued of a given set of polygons. Such constellations (also called "cacti") were enumerated in [3,9,10]. The answer one obtains is n^{k-2}.

Suppose the formula is established for constellations with $\leq c$ faces and let us add one more face. A *polygon* of the constellation is a *neighbor* of the $(c+1)$st face if it has at least one edge in common with this face. A *face* of the constellation is a *neighbor* of the $(c+1)$st face if all its bounding polygons are neighbors of this face. The $(c+1)$st face can have any number $2 \leq m \leq k$ of neighboring polygons and any number $0 \leq d \leq c$ of neighboring faces as in Fig. 4. (Note that a polygon that has only one vertex in common with the face is not considered a neighbor.) By Lemma 1.6, every neighbor face is bounded by exactly two consecutive neighbor polygons.

To construct a constellation with $c+1$ faces we must make the following choices.

1. Choose the numbers m and d.

2. Choose m polygons among k and d faces among c to be the neighbors of the $(c+1)$st face. This gives

$$\binom{k}{m}\binom{c}{d}$$

choices. Denote by $D \subset \{1, \ldots, c\}$ the set of the d neighboring faces.

3. Form the $(c+1)$st face using the m chosen polygons. This is done in the following way. Denote the essential vertices of the face by $V_1, \ldots, V_{p_{c+1}}$. We must describe the colors and the order of the edges that will form the intervals between V_i and V_{i+1} for each i. We claim that such a disposition of edges is uniquely determined once we have (arbitrarily) assigned to each of the m polygons the interval $V_i V_{i+1}$ where its first edge will appear (as we go around the face in the clockwise direction). Indeed, the disposition of the edges can be obtained as follows. (a) For each polygon assigned to the interval $V_i V_{i+1}$ take one edge of its color. (b) Order these edges in the increasing order of colors. (c) In the case if either the biggest color used in $V_{i-1} V_i$ is smaller than the smallest color in the list for $V_i V_{i+1}$ or if there are no polygons assigned to $V_i V_{i+1}$, add, at the beginning of the list of colors for $V_i V_{i+1}$, the last color used in $V_{i-1} V_i$. An example of this algorithm is shown in Fig. 5.

It is easy to see that the vertices V_i are indeed the essential vertices. The uniqueness follows from Lemma 1.5.

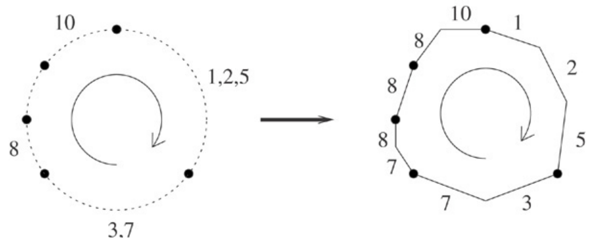


Fig. 5. How to obtain a face from a list of colors assigned to each interval $V_i V_{i+1}$.

Thus the number of ways to form the $(c+1)$st face using m given polygons is p_{c+1}^m.

4. Now we must choose the positions of the d faces that were chosen to be the neighbors of the $(c+1)$st face. There are m "clefts" between the polygons where these faces can be placed. Several faces can appear in the same cleft. In this case they must be ordered (starting from the face closest to the $(c+1)$st one). Thus there are

$$\frac{(d+m-1)!}{(m-1)!}$$

ways to choose the positions of the faces.

5. Each of the neighboring faces has exactly two bounding polygons. We now choose how many edges of the face will belong to the polygon, say, on its left. For the ith face there are p_i choices. Thus we obtain a factor $\prod_{i\in D} p_i$. This product, of course, depends on the particular choice of the d neighboring faces. However, we will soon see (Item 9 and Eq. (1)) that the remaining part of the formula contains the complementary product $\prod_{i\notin D} p_i$, and thus it is not necessary to take a sum over all possible choices of D.

6. We have assembled together all the polygons and faces that are neighbors of the $(c+1)$st face. They form a subconstellation K of the total constellation. Now we are going to *consider K as a unique polygon*. Indeed, we are going to show that the remaining polygons are attached to K in the same way as they would be attached to a unique polygon. In particular, the number of ways to attach them is the same. This will allow us to proceed by induction.

Denote by $r = p_{c+1} + \sum_{i\in D} p_i$ the sum of lengths of the faces of K and by v the total number of vertices of K. These v vertices are acted upon by m cyclic permutation corresponding to the polygons. The product of these m permutations splits the vertices into $d+2$ cycles of lengths p_i $(i \in D)$, p_{c+1}, and $v-r$. The vertices of the last cycle will be called the *essential exterior vertices* of K.

Now consider a constellation R formed by the $k-m$ polygons not used in K and, in addition, one $(v-r)$-gon of color 0. Suppose that the faces of R have lengths p_i, $i \notin D$. We are going to replace the $(v-r)$-gon in R by the constellation K.

Let $p' = p_1 + \cdots + p_c$ and $p = p_1 + \cdots + p_{c+1}$.

7. First we establish a one-to-one correspondence between the vertices of the $(v-r)$-gon and the essential exterior vertices of K preserving their cyclic order. There are $v-r$ ways of doing that, which will account for a factor $v-r$ in the final formula.

In principle, the number $v-r$ is different for different choices of the m polygons and the d faces. However, we will soon see (Eq. (1)) that this number appears in the total sum only as a linear factor. Therefore it will be justifiable *a posteriori* to replace it by its average over the possible choices of m polygons and d faces.

The average value $\langle v - r\rangle$ of $v - r$ equals

$$\frac{m}{k}(n+k+c) - m - d - p_{c+1} - \frac{d}{c}p' = \frac{mn + mc - kp_{c+1}}{k} - d\,\frac{p'+c}{c}.$$

Indeed, the total number of edges in the k polygons equals $n + k + c$ by the Euler formula. Choosing randomly m of the k polygons we obtain an average of $\frac{m}{k}(n + k + c)$ edges in the subconstellation K. Since K has $d + 1$ faces and m polygons, its average number of vertices is $\langle v\rangle = \frac{m}{k}(n + k + c) - m - d$, again by the Euler formula. Now, K has $d + 1$ faces, one of which is always of length p_{c+1} and the other d are chosen randomly from c possibilities. Thus $\langle r\rangle = p_{c+1} + \frac{d}{c}p'$, whence we obtain $\langle v - r\rangle$.

8. Let us go around the exterior face of the constellation K in the counterclockwise direction. The colors of the edges we meet will be increasing in each interval between two consecutive essential exterior vertices. Then, as we pass an essential vertex, the number of the color decreases.

Suppose we are given a new polygon of some color j that does not appear in the constellation K. We want to attach this polygon to the exterior of K in such a way that the cyclic order of colors around each vertex remains increasing (see the remarks after Definition 1.3). It is easy to see that in each interval between two consecutive essential vertices V_i (included) and V_{i+1} (excluded) there is a unique vertex to which the polygon can be attached.

Now, the essential exterior vertices of K are in a one-to-one correspondence with the vertices of a $(v - r)$-gon in the new constellation R. We want to replace the $(v - r)$-gon in R by the subconstellation K. To do this, for each polygon j attached to some vertex of the $(v - r)$-gon, we take the corresponding essential exterior vertex V_i of K and attach our polygon to the unique possible vertex between V_i and V_{i+1}.

It is easy to see that the faces of R, even those that have been modified by our operation, will still have the same number of essential vertices as before. Thus there is a unique way to substitute the $(v - r)$-gon in R by the subconstellation K. The result is the constellation we were trying to assemble.

9. It remains to choose the constellation R. By the induction assumption, there are

$$\frac{(k-m-1+c-d)!}{(k-m-1)!}(n-p)^{k-m-1}\prod_{i\notin D} p_i$$

choices. Indeed, the constellation R has $k - m$ polygons, $c - d$ (interior) faces of lengths p_i, $i \notin D$, and the length of its exterior face is $n - p$ (the same as in the constellation that we are assembling).

4. Computing the sum

The result of our investigation is that the number of constellations (with labeled faces) is given by the following sum:

$$S = p_1 \dots p_c \sum_{m=2}^{k} \binom{k}{m} (n-p)^{k-m-1} p_{c+1}^{m-1} \sum_{d=0}^{c} \binom{c}{d} \frac{(d+m-1)!}{(m-1)!} \times \frac{(k-m-1+c-d)!}{(k-m-1)!} \left(\frac{mn+mc-kp_{c+1}}{k} - d\,\frac{p'+c}{c} \right). \quad (1)$$

The rest of the proof is a sequence of elementary but rather cumbersome computations.

We first compute the subsum over d using the elementary relations

$$\sum_{d=0}^{c} \binom{a+d}{d} \binom{b-d}{c-d} = \binom{a+b+1}{c},$$

$$\sum_{d=0}^{c} d \binom{a+d}{d} \binom{b-d}{c-d} = (a+1) \binom{a+b+1}{c-1}$$

with $a = m-1$, $b = k+c-m-1$. We find that the subsum is equal to

$$\frac{(k+c-1)!}{(k-1)!} \left(\frac{mn+mc-kp_{c+1}}{k} \right) - m \frac{(k+c-1)!}{k!} (p' + c)$$
$$= \frac{(k+c-1)!}{k!} [m(n-p') - kp_{c+1}].$$

Substituting this into the initial sum we obtain

$$S = p_1 \dots p_c \frac{(k+c-1)!}{k!} \sum_{m=2}^{k} \binom{k}{m} (n-p)^{k-m-1} p_{c+1}^{m-1} [m(n-p') - kp_{c+1}].$$

This sum can now be evaluated using two more elementary identities

$$\sum_{m=2}^{k} m \binom{k}{m} x^{k-m-1} y^{m-1} = k \left[\frac{(x+y)^{k-1}}{x} - x^{k-2} \right]$$

$$\sum_{m=2}^{k} \binom{k}{m} x^{k-m-1} y^{m} = \frac{(x+y)^k}{x} - x^{k-1} - kx^{k-2} y,$$

with $x = n-p$, $y = p_{c+1}$. We obtain

$$S = p_1 \dots p_c \frac{(k+c-1)!}{k!} \left\{ k \left[\frac{(n-p')^{k-1}}{n-p} - (n-p)^{k-2} \right] (n-p') \right.$$
$$\left. -k \left[\frac{(n-p')^k}{n-p} - (n-p)^{k-1} - kp_{c+1} (n-p)^{k-2} \right] \right\}$$
$$= p_1 \dots p_c \frac{(k+c-1)!}{(k-1)!} (n-p)^{k-2} \left\{ -(n-p') + (n-p) + kp_{c+1} \right\}$$
$$= p_1 \dots p_{c+1} \frac{(k+c-1)!}{(k-2)!} (n-p)^{k-2}.$$

This is precisely the formula of Theorem 1 with c replaced by $c+1$. The theorem is proved. $\diamond$

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